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CANADA
DOMINION BUREAU OF STATISTICS
CENSUS DIVISION

ANALYTICAL AND TECHNICAL MEMORANDUM

No. 1

PROJECTION OF INCOMPLETE COHORT FERTILITY FOR CANADA BY MEANS OF THE GOMPERTZ FUNCTION

by

A. Romaniuk and S.M. Tanny

Research Sub-Division

Ottawa, March 1969

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PROJECTION OF INCOMPLETE COHORT FERTILITY FOR

CANADA BY MEANS OF THE GOMPERTZ FUNCTION*

Résumé

Pour répondre aux besoins de la projection de la population canadienne actuellement en préparation, cette étude explore les possibilités d'utiliser la fonction de Gompertz pour projeter la fécondité terminale des cohortes de femmes à descendance incomplète. L'application de cette fonction à quelques cohortes à descendance complète, disponibles pour le Canada, produit des résultats encourageants. L'étude reconnaît cependant la nécessité d'améliorer l'estimation des constantes arbitraires de la fonction de Gompertz, ce qui pourrait être tenté par les méthodes de simulation appuyées sur les calculs électroniques.

Abstract

This is an exploratory study to examine the possibility of using the Gompertz function to extrapolate the cohort fertility rates of women who are still in their childbearing ages. The application of this method to the fertility data for the few cohorts of women with completed fertility in Canada gives encouraging results. However, the study indicates the need for improving the technique of estimating arbitrary constants involved in the Gompertz function — a goal that could be reached eventually by means of computer-oriented methods.

1. Introduction

In preparing the 1969 DBS population projections, we are facing the problem of how to project the fertility of the cohorts of women who have not yet completed their childbearing life. We are considering various alternatives for that purpose, but the one we choose to discuss in this paper is the projection of incompleted cumulative cohort fertility rates by means of the Gompertz function.

The Gompertz curve has the shape of a non-symmetrical "S" with a lower '0' asymptote and an upper 'K' asymptote. The curve of the fertility rates by age cumulated through the whole procreative span of a particular birth cohort of women, when plotted against age on arithmetic graph paper, seems to conform closely in shape and behaviour to the theoretical Gompertz curve. It is in the nature of procreative behaviour that the fertility rate, after a slow rise for very young women, increases rapidly, and later as women become older, tends to decrease until

* The authors are indebted to Mr. D. Nagnur of the Research Sub-Division for helpful comments and advice; they are, however, solely responsible for any shortcomings in the paper.

becoming practically nil around age 50. This is true where fertility is mainly a function of the biological factors of procreation; yet the form of the fertility curve remains basically the same where procreation is subjected largely to rational behaviour.

Hence it was felt that it would be worthwhile to try to use the Gompertz function for graduating the actual series of cumulative fertility rates for the Canadian population; by estimating the arbitrary constants involved in the function we might extrapolate for the values of incomplete cohorts. The device has already been applied with encouraging results to the Belgian data by G. Wunsch(1) and to the English, American and Finnish data by P. Martin.(2)

2. Mathematical Statement of the Gompertz Function

In its original form, the Gompertz function is stated as follows:

$$Y = KG^{B^X} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Taking logarithms of both sides of (1) yields

$$\log Y = \log K + B^X \log G \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

For our purposes, it is convenient to alter the form of the function. Thus, we have

$$Y_x = KG^{B^{X-X_o}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

or equivalently,

$$\log Y_x = \log K + B^{X-X_o} \log G \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

and $G > 0;\ 0 < B < 1;\ \log G < 0$

where: Y_x is the expected cumulative fertility at age X ;

K is the upper asymptote, in this case the total fertility rate, that is, the number of children per woman at the end of childbearing age (around 50);

G measures the proportion of children already born to women at age X_o ;

X_o is an arbitrarily chosen but fixed age, and the function attains the maximum as $B^{X-X_o} \rightarrow 0$ (since log G is negative);

log K represents the logarithm of the maximum value;

and B^{X-X_o} represents the factor determining the amount with which the function should tend toward the maximum;

since B is determined to be between 0 and 1 ($0 < B < 1$) we find that B^{X-X_o} is a monotonically decreasing function with increasing x, and hence the function log Y_x attains the maximum with increasing x.

(1) Guillaume Wunsch, "Courbe de Gompertz et perspective de fécondité", Recherches Économiques de Louvain (septembre 1966).

(2) Peter Martin, "Une application des fonctions de Gompertz à l'étude de la fécondité d'une cohorte", Population vol. 22 (novembre-décembre 1967), pp.1085-1096.

3. Solution of the Equation for the Constants K, G and B

Various well-known methods, such as Maximum Likelihood, Least Squares and Optimization Methods cannot easily be modified to handle functions of the Gompertz type. Some approximate methods, that rely on iterative techniques which are ideally suited for computers, are available but are much too cumbersome to be attempted by desk calculators. Simple methods of estimating the arbitrary constants based on a set of selected points have been found to be empirically satisfactory.(3) These points may be points of inflection; often, however, the choice of these points is dependent upon judgement and experience. In particular, the Gompertz curve can be fitted to a set of data by suitable choice of any three points. The calculations may be based entirely on the points chosen, as has been done by Martin, or on the partial totals related to the chosen points, as exemplified by Wunsch. These two techniques are explained here for the sake of making this discussion self-contained.

3.1 The Method of Selected Points

The point X_o and the interval r are arbitrarily chosen. X_1 and X_2 are then determined such that

$$X_2 - X_1 = X_1 - X_o = r \qquad (X_o < X_1 < X_2)$$

Then B, G and K are estimated by solving the equations

$$B^r = \frac{\log Y_2 - \log Y_1}{\log Y_1 - \log Y_o} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

$$\text{or} \quad B = \left(\frac{\log Y_2 - \log Y_1}{\log Y_1 - \log Y_o}\right)^{\frac{1}{r}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5a)$$

$$\log G = \frac{\log Y_1 - \log Y_o}{B^r - 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

$$\log K = \log Y_o - \log G \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

The value of K is also given by:(4)

$$\log K = \frac{(\log Y_o)(\log Y_2) - (\log Y_1)^2}{\log Y_o + \log Y_2 - 2 \log Y_1} \quad \ldots\ldots\ldots\ldots \quad (8)$$

---

(3) See F.E. Croxton and D.L. Cowden, Practical Business Statistics (New York: Prentice-Hall, 1960), Chapter 38; Frederick C. Mills, Statistical Methods (New York: Henry Holt and Co.), Appendix F.

(4) Equation (8) may be derived by substituting equations (5) and (6) into (7), and simplifying.

3.2 The Method of Partial Totals

Instead of selecting points, three partial totals are taken here to provide the basis for the solution of the equation. The observed series is completely subdivided into three equal segments, with r ages in each segment; the sum of log Y_x in each segment is then calculated. We denote these sums by S_o, S_1 and S_2 respectively, with d_1 and d_2 as the first differences, that is

$$S_1 - S_o = d_1$$

$$S_2 - S_1 = d_2$$

We may now solve for B, G and K as follows:

$$B^r = \frac{d_2}{d_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

$$\log G = \frac{d_1\,(B-1)}{(B^r-1)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

$$\log K = \frac{1}{r}\left(S_o - \frac{d_1}{B^r-1}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

Again, if we just wish to derive the value of K, we may use:

$$\log K = \frac{1}{r}\left(S_o - \frac{d_1}{\frac{d_2}{d_1} - 1}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

which is derived by substituting equation (9) into equation (11).

4. Illustrative Examples

We now demonstrate each of the preceding methods on the 1920-21 birth cohort for Canada.

4.1 Method of Selected Points

We choose $X_o = 18, X_1 = 25$ and $X_2 = 32$, with corresponding $Y_o = 72.36$, $Y_1 = 1{,}089$, $Y_2 = 2{,}400$. (We keep four-digit accuracy because of the limitations involved in using four-figure logarithmic tables.) Thus, we find that $\log Y_o = 1.8595$, $\log Y_1 = 3.0370$, $\log Y_2 = 3.3802$.

Using equation (5), we have

$$B^7 = \frac{3.3802 - 3.0370}{3.0370 - 1.8595}$$

$$= 0.2915$$

$$\log B = -.0765$$

$$\therefore \quad B = 0.8385$$

$$\log G = -\frac{3.0370 - 1.8595}{1 - 0.2915} \quad \text{by (6)}$$

$$= -1.6620$$

Thus, using (7), we find that

$$\log K = 1.8595 + 1.6620$$

$$= 3.5215$$

Once we have the values for B, G and K, the function is fully determined and we can estimate the expected cumulative fertility at any desired age; the scheme for calculating may be set up as follows:

TABLE 1. Actual Cohort (Born June 1, 1920 to May 31, 1921)*

$X_o = 18$; $\log B = -.0765$; $\log K = 3.5215$; $r = 7$; $\log G = -1.6620$

Age	$Z=X-X_o$	$Z \log B$	B^Z	$B^Z \log G$	$\log Y$	Y
14	-4	0.3060	2.023	-3.3622	0.1593	1
15	-3	0.2295	1.696	-2.8188	0.7027	5
16	-2	0.1530	1.422	-2.3634	1.1581	14
17	-1	0.0765	1.192	-1.9811	1.5404	35
18	0	0.0000	1.000	-1.6620	1.8595	72
19	1	$\bar{1}.9235$	0.8385	-1.3936	2.1279	134
20	2	$\bar{1}.8470$	0.7031	-1.1686	2.3529	225
21	3	$\bar{1}.7705$	0.5895	-0.9797	2.5418	348
22	4	$\bar{1}.6940$	0.4943	-0.8215	2.7000	501
23	5	$\bar{1}.6175$	0.4145	-0.6889	2.8326	680
24	6	$\bar{1}.5410$	0.3475	-0.5775	2.9440	879
25	7	$\bar{1}.4645$	0.2914	-0.4843	3.0372	1089
26	8	$\bar{1}.3880$	0.2443	-0.4060	3.1155	1305
27	9	$\bar{1}.3115$	0.2048	-0.3404	3.1811	1517
28	10	$\bar{1}.2350$	0.1718	-0.2855	3.2360	1722
29	11	$\bar{1}.1585$	0.1441	-0.2395	3.2820	1914
30	12	$\bar{1}.0820$	0.1208	-0.2008	3.3207	2092
31	13	$\bar{1}.0055$	0.1013	-0.1684	3.3531	2255
32	14	$\bar{2}.9290$	0.0849	-0.1411	3.3804	2401
33	15	$\bar{2}.8525$	0.0712	-0.1183	3.4032	2530
34	16	$\bar{2}.7760$	0.0597	-0.0992	3.4223	2644
35	17	$\bar{2}.6995$	0.0501	-0.0832	3.4383	2744
36	18	$\bar{2}.6230$	0.0420	-0.0698	3.4517	2830
37	19	$\bar{2}.5465$	0.0352	-0.0585	3.4630	2904
38	20	$\bar{2}.4700$	0.0295	-0.0490	3.4725	2968
39	21	$\bar{2}.3935$	0.0248	-0.0411	3.4804	3023
40	22	$\bar{2}.3170$	0.0208	-0.0345	3.4870	3069
41	23	$\bar{2}.2405$	0.0174	-0.0289	3.4926	3109
42	24	$\bar{2}.1640$	0.0146	-0.0242	3.4973	3143
43	25	$\bar{2}.0875$	0.0122	-0.0203	3.5012	3171
44	26	$\bar{2}.0110$	0.0103	-0.0170	3.5044	3195
45	27	$\bar{3}.9345$	0.0086	-0.0143	3.5072	3215
46	28	$\bar{3}.8580$	0.0072	-0.0120	3.5095	3232
47	29	$\bar{3}.7815$	0.0060	-0.0100	3.5115	3247
48	30	$\bar{3}.7050$	0.0051	-0.0084	3.5131	3259
49	31	$\bar{3}.6285$	0.0043	-0.0071	3.5144	3269

* From here on we shall use the notation Actual Cohort (1920-21) to mean Actual Cohort (Born June 1, 1920 to May 31, 1921).

4.2 Method of Partial Totals

We separate the given data into three equal segments, including as much of the data as possible. For the 1920-21 cohort, we use the segments corresponding to the age groups 15-24, 25-34, and 35-44. For this division, $X_o = 15$, i.e., X_o is the first age in the first segment. Thus, we have the following:

$$S_o = \sum_{j=15}^{24} \log Y_j$$

$$= 20.5430$$

$$S_1 = \sum_{j=25}^{34} \log Y_j$$

$$= 32.7304$$

$$S_2 = \sum_{j=35}^{44} \log Y_j$$

$$= 34.9055$$

$$d_1 = S_1 - S_o$$

$$= 12.1874$$

$$d_2 = S_2 - S_1$$

$$= 2.1751$$

$$B^{10} = 0.1785$$

$$\log G = -2.8567$$

$$\log K = 3.5379$$

In the above, B^{10}, log G and log K were calculated using formulas (9), (10), and (11), respectively. Once we have K, G and B, we may set up the same table as we used for selected points to calculate the fertility for all ages between 14 and 49 inclusive. The results of these calculations are presented in Table 2, Column 4.

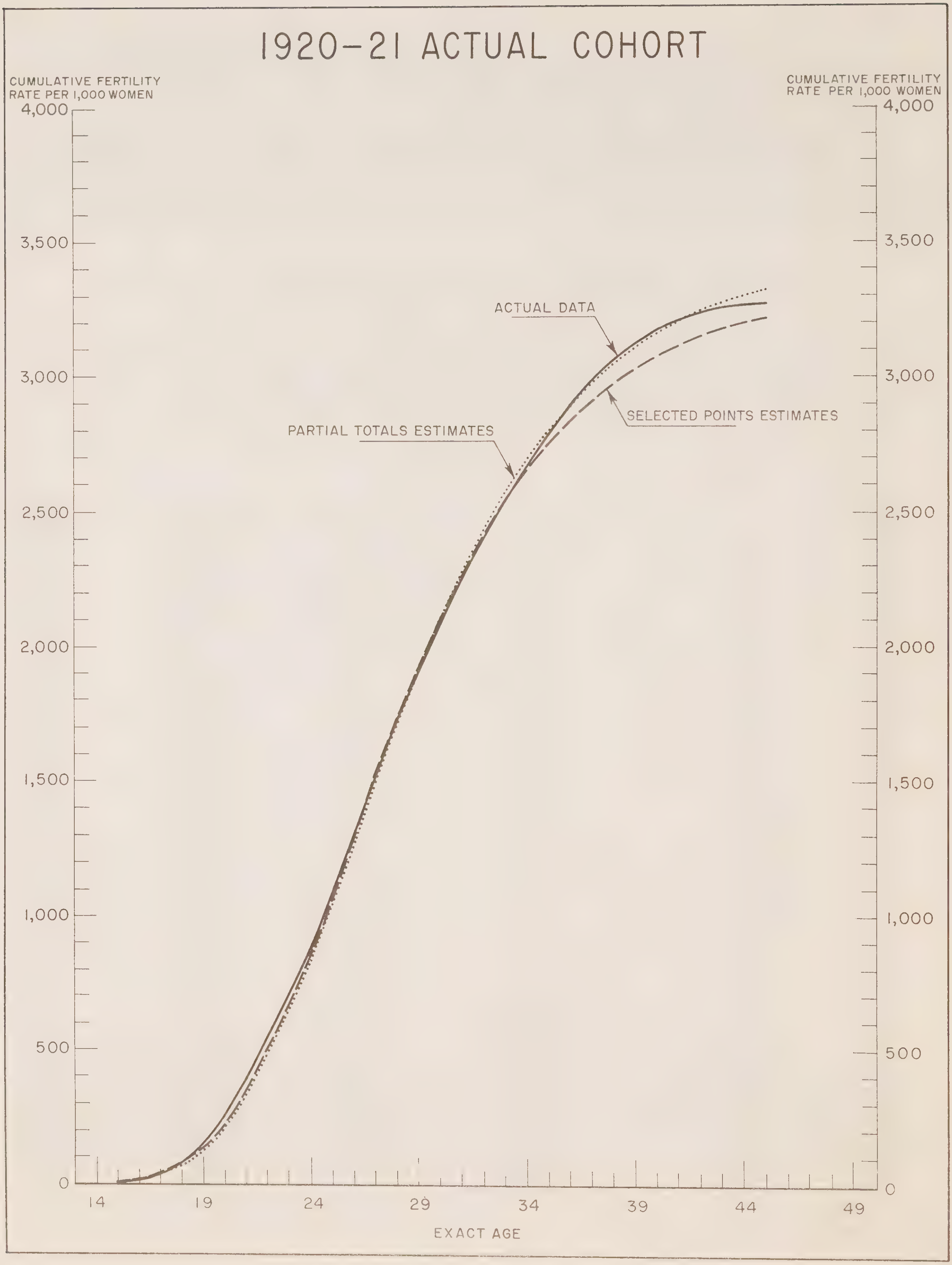
Table 2 shows the actual and graduated values of Y's (cumulative fertility rates) obtained by the two methods as well as the magnitude of the deviations in each case by age. Illustration is provided by Graph 1.

The two methods yield results that deviate in each case in direction and magnitude from the actual series. But it is encouraging to find that the deviations are small except for the younger ages; however these are of little interest for us, since the main objective sought here is the projection of the terminal fertility rate.

TABLE 2. Actual and Graduated Values of Fertility Rates for the Cohort of Women Born Between June 1, 1920 and May 31, 1921

Age	Actual Y values	Expected Y values obtained by:		$\frac{(3)}{(2)} \cdot 100$	$\frac{(4)}{(2)} \cdot 100$
		Selected points method	Partial totals method		
(1)	(2)	(3)	(4)	(5)	(6)
15	3	5	5	166.67	166.67
16	11	14	14	127.27	127.27
17	31	35	33	112.90	106.45
18	72	72	68	100.00	94.44
19	145	134	127	92.41	87.59
20	251	225	214	89.64	85.26
21	388	348	332	89.69	85.57
22	545	501	481	91.93	88.26
23	709	680	657	95.91	92.67
24	884	879	855	99.43	96.72
25	1089	1089	1066	100.00	97.89
26	1314	1305	1283	99.31	97.64
27	1516	1517	1501	100.07	99.01
28	1715	1722	1713	100.41	99.88
29	1900	1914	1913	100.74	100.68
30	2078	2092	2100	100.67	101.06
31	2240	2255	2272	100.67	101.43
32	2400	2401	2427	100.04	101.12
33	2545	2530	2565	99.41	100.79
34	2674	2644	2689	98.88	100.56
35	2789	2744	2797	98.39	100.29
36	2895	2830	2892	97.75	99.90
37	2984	2904	2974	97.32	99.66
38	3062	2968	3045	96.93	99.44
39	3125	3023	3106	96.74	99.39
40	3174	3069	3158	96.69	99.50
41	3208	3109	3202	96.91	99.81
42	3234	3143	3241	97.19	100.22
43	3251	3171	3273	97.54	100.68
44	3260	3195	3300	98.01	101.23
45	3264	3215	3324	98.50	101.84

GRAPH 1


1920-21 ACTUAL COHORT
CUMULATIVE FERTILITY RATE PER 1,000 WOMEN
CUMULATIVE FERTILITY RATE PER 1,000 WOMEN
4,000
3,500
3,000
2,500
2,000
1,500
1,000
500
0
ACTUAL DATA
PARTIAL TOTALS ESTIMATES
SELECTED POINTS ESTIMATES
14
19
24
29
34
39
44
49
EXACT AGE

5. The Selection of the Method

It is not easy to decide which one of the two methods illustrated above is likely to yield a better fit and hence would permit to express the expected trend of fertility more satisfactorily. The method of partial totals may be considered as being more objective than the method of selected points, but the latter is more flexible because it can be easily adapted so as to yield a closer fit for particular segments of the curve in which we are interested. It also has the advantage of greater computational ease than the other method.

The ultimate determining criterion of choice in this particular situation was the relative efficiency of the two methods in faithfully reproducing that portion of the cumulative fertility curve, the values of which we intended to extrapolate. From our pilot investigations based on numerous trials we found that the method of selected points suited our purpose better since the curve representing the final phases of cumulative fertility experience could more closely be reflected by the choice of a suitable set of points. However, certain restrictive criteria had to be adhered to before the set of points was selected:

(i) data representing age 14 were regarded as unreliable and had to be left out; and for similar reasons ages 15, 16 and 17 were excluded unless 'r' was large, say over 13;

(ii) the spread between the points (r) was kept as large as possible in order to include as much of the fertility experience as possible.

Keeping in mind the above restrictions, we experimented with several sets of points (such as ages 18, 24, 30; 18, 25, 32; 20, 30, 40, etc.) and found that the set 18, 25 and 32 reflected the experience closer than other sets. Hence, this set was used to extrapolate the cumulative fertility for ages 33 and over. However, for graduating the rates for the cohorts with completed fertility, the points used were the ages 18, 33 and 48.

6. Graduation of the Cumulative Fertility Rates

For graduating purposes three actual(5) birth cohorts and two hypothetical(5) cohorts of women were used. Since we desired cohorts with completed fertility, our choice of actual cohorts was by necessity restricted to early ones, and we therefore chose the cohorts of 1914-15, 1915-16, and 1916-17.

---

(5) The actual cohort fertility represents the fertility experience of a group of women born during a given period (in this case between June 1,and May 31 of the respective years) through their entire procreative span of life. However, it should be pointed out that fertility by birth cohorts of women used here is based on data abstracted from the current vital statistics with some adaptations. The procedure followed in constructing the cohort fertility rates will be explained in a forthcoming memorandum; the point has no direct relation to the topic of the present paper.

Hypothetical cohort fertility represents the childbearing experience of women during a particular year or specified period of time. Thus, it is obtained by cumulating the current age-specific fertility rates. The rates are measuring the childbearing performance at a given moment of a whole range of birth cohorts of women and not of a single cohort of women as in the previous case.

While a wide range of hypothetical cohorts was available only two of them were chosen, those of 1950 and 1962, mainly because our interest in these hypothetical cohorts had been purely experimental.

The results of the graduations appear in Tables A and B, while graphical illustrations are provided in Graphs A.1 to A.5.

There is a distinct pattern in the results of the graduations. For ages 14-17 inclusive, the graduated values are much higher than the actual ones; at ages 18, 33 and 48 the observed and the expected values coincide; finally, for the last two of the three given ages, the graduated values are slightly below the actual values. In most cases, the deviation between the graduated and actual values is less than 4%.

The method seems to yield a better graduation for actual cohort data than for hypothetical cohort data. While the number of tested series is too small to draw a definitive conclusion on this score, from a logical point of view the result is not unexpected. Indeed, the genuine shape of the cumulative fertility by age is probably more forcefully expressed by the fertility of actual cohorts than by that based on the age-specific fertility rates at a particular moment, which constitute the basis of the hypothetical cumulative fertility cohort. As a result of the temporary fluctuation in fertility at a particular moment, the latter may more or less strongly depart from the "typical" age pattern of fertility of a given birth cohort of women.

7. Projection of the Remaining Cohort Fertility

As has already been mentioned, the method of selected points was chosen to complete the cohorts from 1917-18 to 1933-34. The ages used were 18, 25 and 32. The choice of these ages was based on empirical considerations — this set was found to yield a "good" asymptote (i.e. K value) consistently, in the sense that the calculated asymptote and the cumulative fertility at age 45(6) consistently showed a small positive deviation. Naturally, this is what we desire, since it is the latter portion of the curve which interests us here.

The results of these projections (see Appendix C) bear close inspection. To start with, there is ample evidence to suggest that the projections for the cohorts 1917-18 through to 1927-28 are too low.(7) This conclusion is drawn by comparing, where possible, the actual and projected figures for ages 33 through 48. In 120 of 121 cases, the actual figure was higher than the projected one. However, except for the 1918-19 cohort, the actual figure never exceeds the projected one by more than 4.5% and generally by less than 2.5%. Another interesting result is the pattern in the algebraic sign of the difference (projected value — actual value). In almost every case, the difference is positive from ages 14-17 inclusive, negative through ages 19-24, positive for ages 26-31, and negative from age 33 onward. That is, the selected points seem to act as a pivot for the sign of this difference.

(6) Age 45 rather than 49 was used because more cohorts could be made available for testing. This does not cause any significant discrepancy as the cohort fertility is essentially complete by age 45.

(7) There is not enough evidence to draw any conclusions for the cohorts from 1928-29 to 1933-34.

The important fact for us, however, is that after age 32, the projected values seem to be too low for the first eleven cohorts and it is a fair assumption that the same pattern holds good for the later cohorts. For this reason a correction factor was applied to the values corresponding to ages 33 to 49 to render these values a little higher and more realistic.(8)

8. Concluding Remarks

The results arrived at in this paper should be considered with the limitations reported earlier; the test of the method lies in its empirical validity. For the data under consideration, the method employed may be deemed fairly satisfactory. It must be kept in mind however, that for the method to be universally applicable, better techniques to estimate the arbitrary constants would be required. Such an investigation could be effectively reinforced by the use of computer-oriented methods.

(8) See the Appendix A for a discussion of the correction applied to the data.

TABLE A. Graduation of Cumulative Fertility Rates (per 1,000 Women)

Age	1914-15 actual cohort			1915-16 actual cohort			1916-17 actual cohort		
	Actual rate	Graduated rate	% deviation	Actual rate	Graduated rate	% deviation	Actual rate	Graduated rate	% deviation
14	1	2	+ 100.00	1	2	+ 100.00	1	2	+ 100.00
15	3	7	+ 133.33	3	6	+ 100.00	3	6	+ 100.00
16	11	18	+ 63.64	11	16	+ 45.45	11	16	+ 45.45
17	32	39	+ 21.88	31	36	+ 16.13	30	35	+ 16.67
18	76	76	10.00	71	70	− 1.41	70	70	—
19	141	132	− 6.38	135	105	− 7.41	132	125	− 5.30
20	224	212	− 5.36	216	203	− 6.01	213	204	− 4.23
21	326	317	− 2.76	318	307	− 3.46	320	309	− 3.44
22	445	445	0.00	445	434	− 2.47	446	428	− 1.79
23	587	594	+ 1.19	583	586	+ 0.51	593	589	− 0.67
24	736	758	+ 2.99	739	766	+ 3.65	754	756	+ 0.27
25	898	934	+ 4.01	904	924	+ 2.21	922	933	+ 1.19
26	1,066	1,114	+ 4.50	1,076	1,105	+ 2.70	1,098	1,115	+ 1.55
27	1,237	1,293	+ 4.53	1,252	1,285	+ 2.64	1,265	1,296	+ 2.45
28	1,421	1,469	+ 3.38	1,426	1,461	+ 2.45	1,434	1,472	+ 2.65
29	1,589	1,635	+ 2.89	1,582	1,628	+ 2.91	1,605	1,640	+ 2.18
30	1,747	1,793	+ 2.63	1,752	1,784	+ 1.83	1,776	1,796	+ 1.13
31	1,899	1,936	+ 1.95	1,904	1,928	+ 1.26	1,917	1,939	+ 1.15
32	2,053	2,056	+ 0.15	2,045	2,059	+ 0.68	2,057	2,067	+ 0.49
33	2,187	2,186	− 0.05	2,176	2,177	+ 0.05	2,185	2,185	—
34	2,308	2,293	− 0.65	2,294	2,282	− 0.52	2,300	2,290	− 0.43
35	2,416	2,387	− 1.20	2,400	2,374	− 1.08	2,403	2,381	− 0.92
36	2,515	2,470	− 1.39	2,497	2,456	− 1.64	2,499	2,461	− 1.52
37	2,603	2,543	− 2.31	2,585	2,527	− 2.24	2,587	2,530	− 2.20
38	2,683	2,606	− 2.87	2,665	2,589	− 2.85	2,666	2,591	− 2.81
39	2,750	2,661	− 3.24	2,730	2,642	− 3.22	2,728	2,637	− 3.34
40	2,803	2,708	− 3.39	2,780	2,688	− 3.21	2,778	2,688	− 3.24
41	2,840	2,749	− 3.21	2,816	2,729	− 3.09	2,812	2,727	− 3.02
42	2,871	2,785	− 3.00	2,845	2,763	− 2.88	2,840	2,760	− 2.82
43	2,890	2,815	− 2.60	2,863	2,792	− 2.48	2,858	2,788	− 2.45
44	2,901	2,841	− 2.07	2,874	2,811	− 2.02	2,869	2,813	− 1.95
45	2,908	2,863	− 1.55	2,880	2,838	− 1.46	2,874	2,832	− 1.46
46	2,911	2,882	− 1.50	2,883	2,856	− 0.94	2,877	2,850	− 0.94
47	2,912	2,898	− 0.48	2,884	2,872	− 0.42	2,878	2,865	− 0.45
48	2,912	2,912	0.00	2,884	2,885	+ 0.03	2,878	2,878	—
49	2,912	2,924	+ 0.47	2,885	2,896	+ 0.38	2,879	2,889	+ 0.35

TABLE A. Graduation of Cumulative Fertility Rates (per 1,000 Women) — Concluded

Age	Hypothetical cohort (1950)			Hypothetical cohort (1962)		
	Actual rate	Graduated rate	% deviation	Actual rate	Graduated rate	% deviation
14	1	4	+ 300.00	1	5	+ 400.00
15	4	12	+ 200.00	6	14	+ 133.33
16	18	30	+ 66.66	24	36	+ 50.00
17	52	64	+ 2.31	67	80	+ 12.40
18	121	121	0.00	154	153	— 0.65
19	228	206	— 9.65	293	265	— 9.56
20	370	323	— 12.70	476	416	— 12.61
21	537	471	— 12.29	702	605	— 13.82
22	724	647	— 10.64	945	827	— 12.49
23	924	845	— 8.55	1,199	1,071	— 10.68
24	1,131	1,059	— 6.37	1,452	1,328	— 8.54
25	1,343	1,299	— 4.77	1,690	1,569	— 5.98
26	1,551	1,500	— 3.29	1,923	1,843	— 4.16
27	1,747	1,716	— 1.77	2,138	2,084	— 2.53
28	1,946	1,921	— 1.28	2,341	2,310	— 1.32
29	2,131	2,113	— 0.84	2,526	2,514	— 0.48
30	2,306	2,288	— 0.78	2,701	2,699	— 0.07
31	2,447	2,446	— 0.04	2,852	2,863	+ 0.39
32	2,588	2,590	+ 0.08	3,006	3,005	— 0.03
33	2,716	2,716	0.00	3,130	3,130	0.00
34	2,834	2,827	— 0.21	3,243	3,237	— 0.19
35	2,942	2,925	— 0.58	3,340	3,329	— 0.33
36	3,043	3,069	+ 1.12	3,429	3,408	— 0.61
37	3,129	3,082	— 1.50	3,507	3,473	— 0.99
38	3,206	3,145	— 1.90	3,573	3,530	— 1.20
39	3,269	3,199	— 2.14	3,629	3,578	— 1.41
40	3,320	3,244	— 2.29	3,675	3,617	— 1.58
41	3,356	3,283	— 2.18	3,710	3,651	— 1.59
42	3,387	3,316	— 2.10	3,737	3,679	— 1.55
43	3,407	3,345	— 1.82	3,754	3,701	— 1.41
44	3,419	3,369	— 1.46	3,763	3,721	— 1.12
45	3,426	3,389	— 1.08	3,760	3,738	— 0.80
46	3,430	3,406	— 0.70	3,771	3,752	— 0.50
47	3,432	3,422	— 0.29	3,773	3,762	— 0.29
48	3,433	3,434	+ 0.03	3,773	3,772	— 0.03
49	3,433	3,443	+ 0.29	3,773	3,780	+ 0.19

TABLE B. Projections of Cumulative Fertility Rates (per 1,000 Women)

Age	1917-18 actual cohort			1918-19 actual cohort			1919-20 actual cohort			1920-21 actual cohort		
	Actual rate	Projected rate		Actual rate	Projected rate		Actual rate	Projected rate		Actual rate	Projected rate	
		Original	Corrected		Original	Corrected		Original	Corrected		Original	Corrected
14	0	2		1	1		0	2		1	1	
15	3	5		3	4		3	6		3	5	
16	9	14		10	13		11	16		11	14	
17	29	33		28	32		30	36		31	35	
18	68	68		67	67		74	74		72	72	
19	129	123		129	125		142	135		145	134	
20	215	203		213	211		242	223		251	225	
21	321	310		325	325		374	341		388	348	
22	459	443		465	466		533	488		545	501	
23	614	597		620	629		703	659		709	680	
24	778	769		786	807		876	850		884	879	
25	950	950		949	994		1,053	1,054		1,089	1,089	
26	1,122	1,137		1,114	1,180		1,262	1,262		1,314	1,305	
27	1,288	1,322		1,296	1,363		1,475	1,470		1,516	1,517	
28	1,473	1,500		1,487	1,536		1,673	1,670		1,715	1,722	
29	1,650	1,669		1,651	1,696		1,857	1,862		1,900	1,914	
30	1,813	1,826		1,810	1,842		2,032	2,039		2,078	2,092	
31	1,958	1,970		1,951	1,973		2,189	2,203		2,240	2,255	
32	2,099	2,099		2,091	2,089		2,348	2,349		2,400	2,401	
33	2,227	2,215	2,255	2,219	2,192	2,219	2,489	2,480	2,528	2,545	2,530	2,572
34	2,346	2,317	2,359	2,335	2,280	2,308	2,621	2,597	2,647	2,674	2,644	2,688
35	2,453	2,407	2,450	2,441	2,357	2,386	2,740	2,701	2,753	2,789	2,744	2,789
36	2,552	2,486	2,531	2,535	2,422	2,452	2,844	2,789	2,843	2,895	2,830	2,877
37	2,640	2,554	2,600	2,620	2,478	2,509	2,937	2,867	2,922	2,984	2,904	2,952
38	2,718	2,613	2,660	2,695	2,525	2,556	3,016	2,934	2,991	3,062	2,968	3,017
39	2,781	2,663	2,711	2,754	2,566	2,598	3,079	2,992	3,050	3,125	3,023	3,073
40	2,830	2,706	2,755	2,800	2,600	2,632	3,131	3,042	3,101	3,174	3,069	3,120
41	2,864	2,743	2,792	2,832	2,628	2,661	3,167	3,084	3,144	3,208	3,109	3,160
42	2,892	2,774	2,824	2,858	2,653	2,686	3,195	3,121	3,181	3,234	3,143	3,195
43	2,909	2,802	2,852	2,875	2,673	2,706	3,212	3,152	3,213	3,251	3,171	3,223
44	2,919	2,824	2,875	2,884	2,689	2,722	3,222	3,178	3,239	3,260	3,195	3,248
45	2,924	2,843	2,894	2,889	2,704	2,738	3,228	3,200	3,262	3,264	3,215	3,268
46	2,927	2,860	2,911	2,892	2,715	2,749	3,230	3,220	3,282		3,232	3,285
47	2,928	2,874	2,926	2,893	2,726	2,760		3,235	3,297		3,247	3,301
48	2,928	2,885	2,937		2,733	2,767		3,248	3,311		3,259	3,313
49		2,896	2,948		2,740	2,774		3,260	3,323		3,269	3,323

TABLE B. Projections of Cumulative Fertility Rates (per 1,000 Women) — Continued

Age	1921-22 actual cohort			1922-23 actual cohort			1923-24 actual cohort			1924-25 actual cohort		
	Actual rate	Projected rate		Actual rate	Projected rate		Actual rate	Projected rate		Actual rate	Projected rate	
		Original	Corrected		Original	Corrected		Original	Corrected		Original	Corrected
14	1	1		0	1		1	1		0	1	
15	3	5		3	5		2	5		3	5	
16	11	14		10	14		10	14		11	15	
17	30	36		30	35		31	37		33	37	
18	75	75		75	75		79	79		81	81	
19	153	141		154	142		158	149		157	154	
20	265	238		261	240		261	253		261	262	
21	401	369		386	373		390	393		415	406	
22	552	530		532	537		564	564		611	583	
23	717	718		723	726		778	762		814	787	
24	922	925		942	934		987	977		1,025	1,008	
25	1,145	1,141		1,151	1,151		1,200	1,201		1,238	1,237	
26	1,356	1,361		1,356	1,370		1,407	1,424		1,448	1,466	
27	1,558	1,576		1,553	1,581		1,603	1,641		1,654	1,686	
28	1,758	1,780		1,745	1,782		1,803	1,844		1,857	1,892	
29	1,942	1,972		1,931	1,969		1,993	2,032		2,051	2,083	
30	2,123	2,146		2,113	2,140		2,176	2,203		2,234	2,254	
31	2,287	2,304		2,273	2,292		2,335	2,354		2,389	2,408	
32	2,446	2,445		2,427	2,427		2,486	2,488		2,540	2,539	
33	2,588	2,568	2,604	2,562	2,545	2,575	2,623	2,604	2,632	2,675	2,656	2,683
34	2,716	2,676	2,713	2,686	2,646	2,677	2,746	2,704	2,733	2,798	2,755	2,783
35	2,831	2,770	2,809	2,795	2,735	2,767	2,855	2,790	2,820	2,905	2,841	2,870
36	2,932	2,849	2,889	2,892	2,809	2,842	2,948	2,864	2,895	2,998	2,913	2,942
37	3,020	2,918	2,959	2,973	2,874	2,908	3,029	2,926	2,958	3,075	2,975	3,005
38	3,095	2,977	3,018	3,044	2,929	2,963	3,095	2,979	3,011	3,141	3,026	3,057
39	3,154	3,027	3,069	3,100	2,975	3,010	3,149	3,023	3,056	3,193	3,069	3,100
40	3,200	3,069	3,112	3,142	3,014	3,049	3,191	3,060	3,093	3,230	3,106	3,137
41	3,232	3,105	3,148	3,173	3,046	3,082	3,218	3,091	3,125	3,253	3,136	3,168
42	3,257	3,134	3,178	3,194	3,080	3,116	3,236	3,117	3,151		3,161	3,193
43	3,271	3,159	3,203	3,207	3,096	3,132		3,138	3,172		3,182	3,214
44	3,279	3,181	3,225		3,116	3,152		3,156	3,190		3,200	3,232
45		3,199	3,243		3,139	3,176		3,172	3,207		3,215	3,247
46		3,214	3,259		3,145	3,182		3,184	3,219		3,226	3,259
47		3,226	3,271		3,156	3,193		3,196	3,231		3,237	3,270
48		3,237	3,282		3,165	3,202		3,203	3,238		3,245	3,278
49		3,245	3,290		3,174	3,211		3,210	3,245		3,252	3,285

TABLE B. Projections of Cumulative Fertility Rates (per 1,000 Women) — Continued

Age	1925-26 actual cohort			1926-27 actual cohort			1927-28 actual cohort			1928-29 actual cohort		
	Actual rate	Projected rate		Actual rate	Projected rate		Actual rate	Projected rate		Actual rate	Projected rate	
		Original	Corrected		Original	Corrected		Original	Corrected		Original	Corrected
14	0	1		1	1		1	1		1	1	
15	2	4		3	4		3	5		3	6	
16	11	14		11	13		12	15		12	18	
17	33	36		32	35		34	40		37	46	
18	79	79		79	79		88	88		98	98	
19	155	153		167	154		191	169		199	184	
20	278	262		308	266		332	288		337	308	
21	452	408		472	417		500	446		505	471	
22	639	590		659	603		687	640		699	669	
23	841	798		858	816		895	861		916	892	
24	1,047	1,023		1,067	1,047		1,117	1,097		1,141	1,129	
25	1,256	1,256		1,283	1,283		1,341	1,340		1,371	1,372	
26	1,472	1,488		1,496	1,517		1,563	1,579		1,594	1,610	
27	1,682	1,710		1,709	1,740		1,778	1,806		1,812	1,837	
28	1,888	1,918		1,914	1,947		1,990	2,017		2,023	2,047	
29	2,077	2,108		2,103	2,135		2,182	2,209		2,213	2,239	
30	2,258	2,227		2,282	2,304		2,359	2,379		2,393	2,408	
31	2,414	2,430		2,435	2,452		2,516	2,529		2,545	2,559	
32	2,562	2,563		2,581	2,581		2,660	2,660		2,688	2,689	
33	2,695	2,676	2,701	2,711	2,693	2,715	2,788	2,771	2,793	2,813	2,801	2,824
34	2,814	2,774	2,780	2,824	2,786	2,808	2,901	2,867	2,890	2,922	2,897	2,921
35	2,915	2,858	2,884	2,921	2,866	2,889	2,999	2,947	2,971	3,014	2,979	3,004
36	3,004	2,927	2,954	3,008	2,934	2,957	3,082	3,016	3,040	3,088	3,047	3,072
37	3,078	2,987	3,014	3,080	2,989	3,013	3,147	3,073	3,098	3,143	3,105	3,131
38	3,141	3,035	3,063	3,138	3,037	3,061	3,196	3,120	3,145		3,153	3,179
39	3,187	3,079	3,107	3,178	3,076	3,101		3,160	3,186		3,193	3,220
40	3,219	3,114	3,143		3,109	3,134		3,193	3,219		3,226	3,253
41		3,143	3,172		3,135	3,160		3,221	3,247		3,255	3,282
42		3,166	3,195		3,158	3,183		3,243	3,269		3,277	3,304
43		3,186	3,215		3,177	3,202		3,262	3,288		3,297	3,324
44		3,203	3,232		3,192	3,218		3,277	3,304		3,313	3,340
45		3,217	3,247		3,204	3,230		3,291	3,318		3,326	3,354
46		3,228	3,258		3,215	3,241		3,301	3,328		3,336	3,364
47		3,238	3,268		3,223	3,249		3,309	3,336		3,345	3,373
48		3,245	3,275		3,229	3,255		3,316	3,343		3,353	3,381
49		3,252	3,282		3,236	3,262		3,322	3,349		3,359	3,387

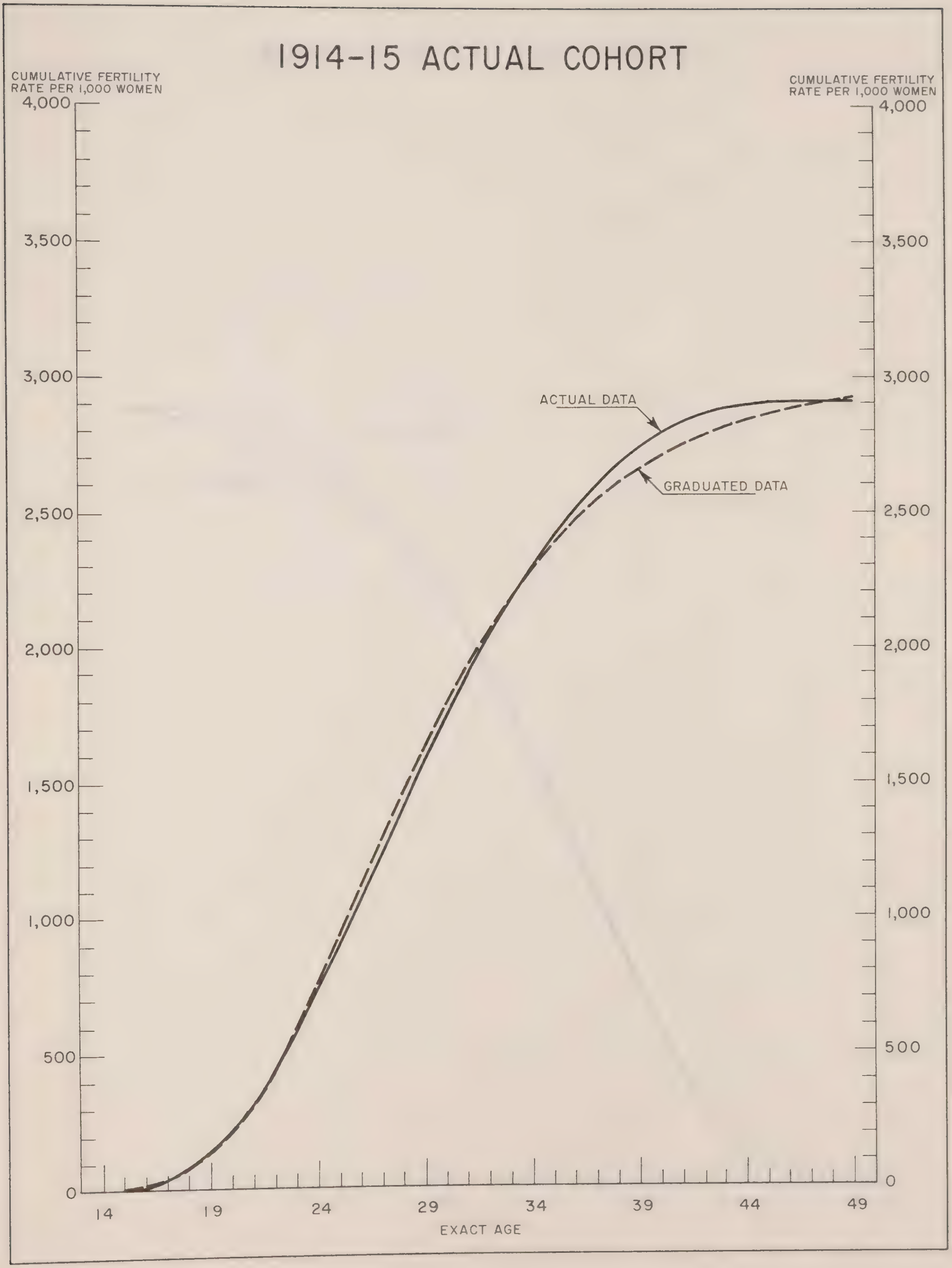
TABLE B. Projections of Cumulative Fertility Rates (per 1,000 Women) — Continued

Age	1929-30 actual cohort			1930-31 actual cohort			1931-32 actual cohort			1932-33 actual cohort		
	Actual rate	Projected rate		Actual rate	Projected rate		Actual rate	Projected rate		Actual rate	Projected rate	
		Original	Corrected		Original	Corrected		Original	Corrected		Original	Corrected
14	1	2		1	2		1	2		1	2	
15	3	7		3	7		4	7		4	7	
16	13	20		14	21		16	21		17	23	
17	43	51		46	54		49	55		52	58	
18	108	108		114	114		118	118		124	124	
19	214	201		221	212		229	221		244	232	
20	356	334		371	353		388	368		405	384	
21	534	508		563	535		588	559		609	533	
22	744	717		782	753		817	786		835	809	
23	975	952		1,020	996		1,061	1,038		1,081	1,060	
24	1,217	1,201		1,265	1,253		1,314	1,302		1,334	1,322	
25	1,458	1,456		1,513	1,513		1,567	1,566		1,581	1,582	
26	1,695	1,706		1,753	1,765		1,809	1,822		1,821	1,831	
27	1,923	1,944		1,977	2,003		2,040	2,060		2,046	2,060	
28	2,138	2,163		2,195	2,221		2,256	2,277		2,254	2,269	
29	2,338	2,363		2,394	2,418		2,450	2,471		2,440	2,453	
30	2,525	2,541		2,576	2,593		2,625	2,642		2,609	2,614	
31	2,685	2,698		2,733	2,746		2,777	2,789		2,750	2,754	
32	2,834	2,833		2,877	2,877		2,917	2,916		2,871	2,872	
33	2,962	2,949	2,973	3,000	2,989	3,010	3,026	3,024	3,043	2,966	2,972	2,987
34	3,072	3,050	3,075	3,098	3,084	3,106	3,112	3,115	3,134		3,055	3,070
35	3,160	3,134	3,160	3,173	3,165	3,187		3,190	3,210		3,125	3,141
36	3,228	3,206	3,232		3,232	3,255		3,253	3,273		3,183	3,199
37		3,266	3,293		3,291	3,314		3,306	3,326		3,230	3,246
38		3,316	3,343		3,336	3,360		3,350	3,371		3,270	3,286
39		3,358	3,386		3,376	3,400		3,386	3,407		3,302	3,319
40		3,393	3,421		3,408	3,432		3,415	3,436		3,329	3,346
41		3,422	3,450		3,435	3,459		3,440	3,461		3,351	3,368
42		3,446	3,474		3,457	3,482		3,459	3,480		3,368	3,385
43		3,466	3,494		3,475	3,500		3,475	3,497		3,383	3,400
44		3,482	3,511		3,490	3,515		3,489	3,511		3,394	3,411
45		3,496	3,525		3,502	3,527		3,500	3,522		3,404	3,421
46		3,508	3,537		3,514	3,539		3,510	3,532		3,412	3,429
47		3,518	3,547		3,522	3,547		3,517	3,539		3,419	3,436
48		3,525	3,554		3,529	3,554		3,522	3,544		3,424	3,441
49		3,531	3,560		3,534	3,559		3,527	3,549		3,429	3,446

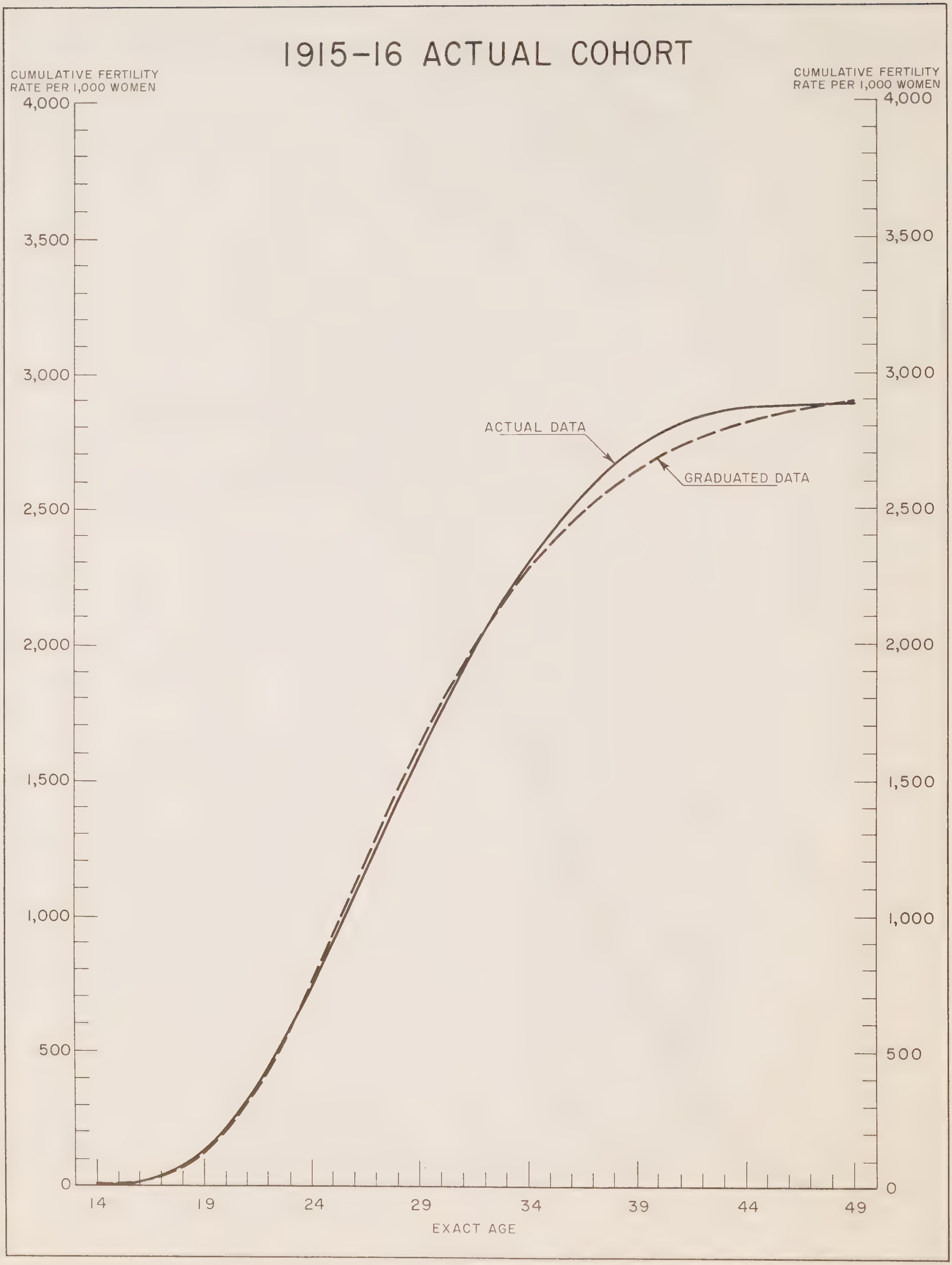
TABLE B. Projections of Cumulative Fertility Rates (per 1,000 Women) — Concluded

Age	1933-34 actual cohort		
	Actual rate	Projected rate	
		Original	Corrected
14	1	2	
15	4	7	
16	18	23	
17	53	59	
18	128	128	
19	249	238	
20	418	393	
21	620	591	
22	841	821	
23	1,088	1,072	
24	1,335	1,329	
25	1,582	1,582	
26	1,815	1,822	
27	2,032	2,043	
28	2,235	2,241	
29	2,415	2,414	
30	2,576	2,564	
31	2,700	2,694	
32	2,801	2,802	
33		2,893	2,906
34		2,969	2,982
35		3,032	3,046
36		3,084	3,098
37		3,126	3,140
38		3,161	3,175
39		3,190	3,204
40		3,213	3,227
41		3,232	3,247
42		3,247	3,262
43		3,260	3,275
44		3,271	3,286
45		3,278	3,293
46		3,285	3,300
47		3,291	3,306
48		3,295	3,310
49		3,298	3,313

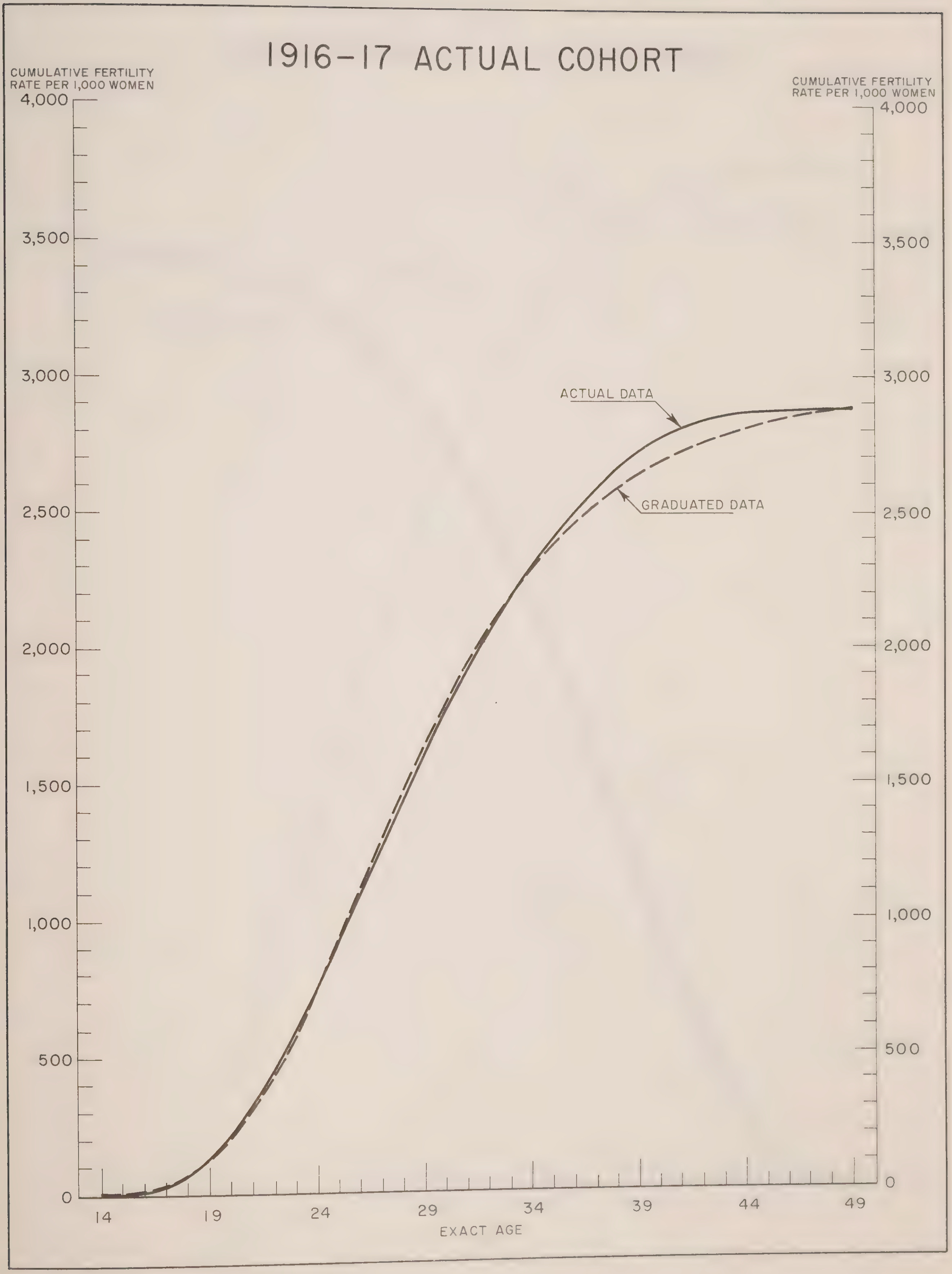
GRAPH A.1


1914-15 ACTUAL COHORT
CUMULATIVE FERTILITY RATE PER 1,000 WOMEN
4,000
3,500
3,000
2,500
2,000
1,500
1,000
500
0
CUMULATIVE FERTILITY RATE PER 1,000 WOMEN
4,000
3,500
3,000
2,500
2,000
1,500
1,000
500
0
ACTUAL DATA
GRADUATED DATA
14
19
24
29
34
39
44
49
EXACT AGE

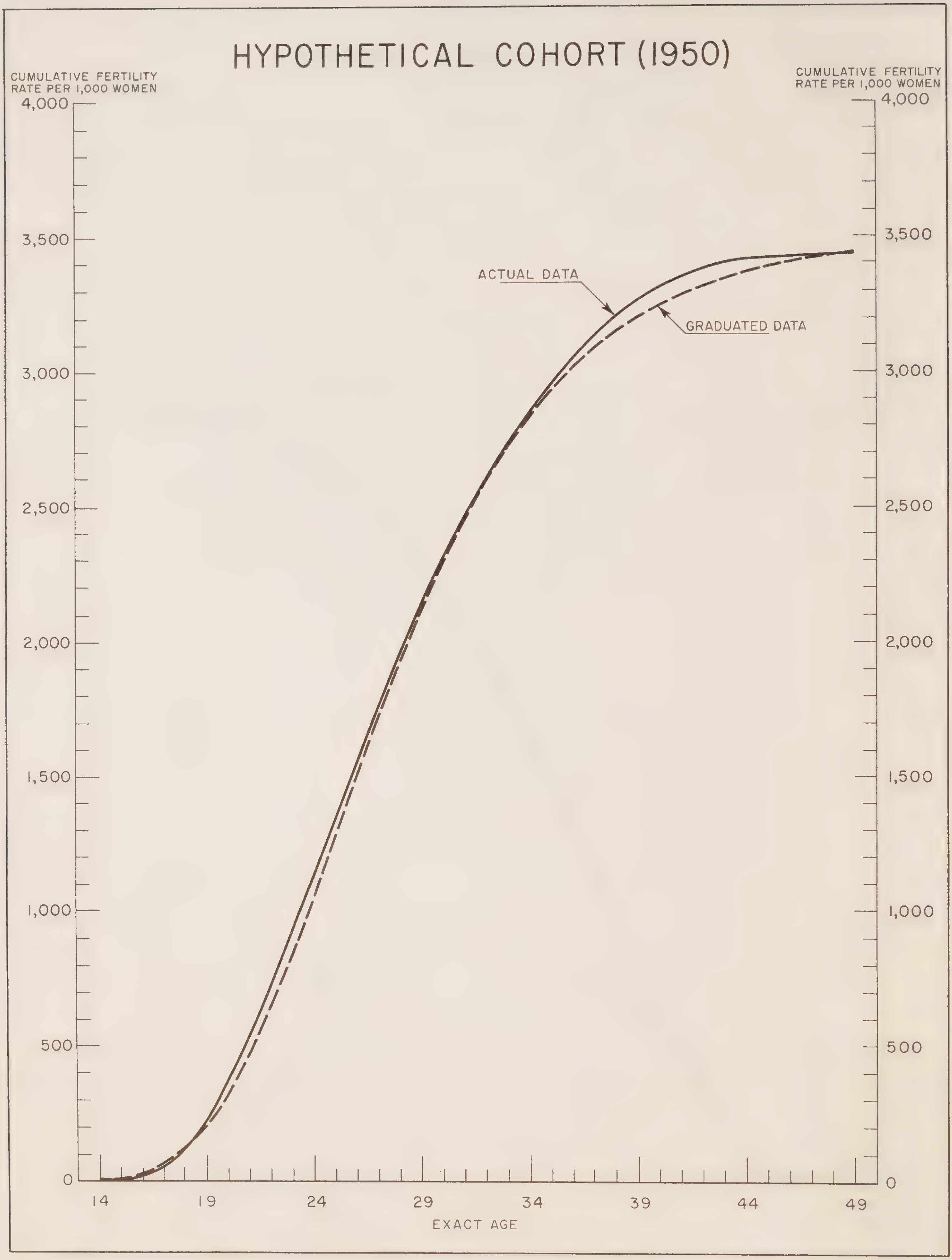
GRAPH A. 2


1915-16 ACTUAL COHORT
CUMULATIVE FERTILITY RATE PER 1,000 WOMEN
CUMULATIVE FERTILITY RATE PER 1,000 WOMEN
4,000
3,500
3,000
2,500
2,000
1,500
1,000
500
0
ACTUAL DATA
GRADUATED DATA
14
19
24
29
34
39
44
49
EXACT AGE

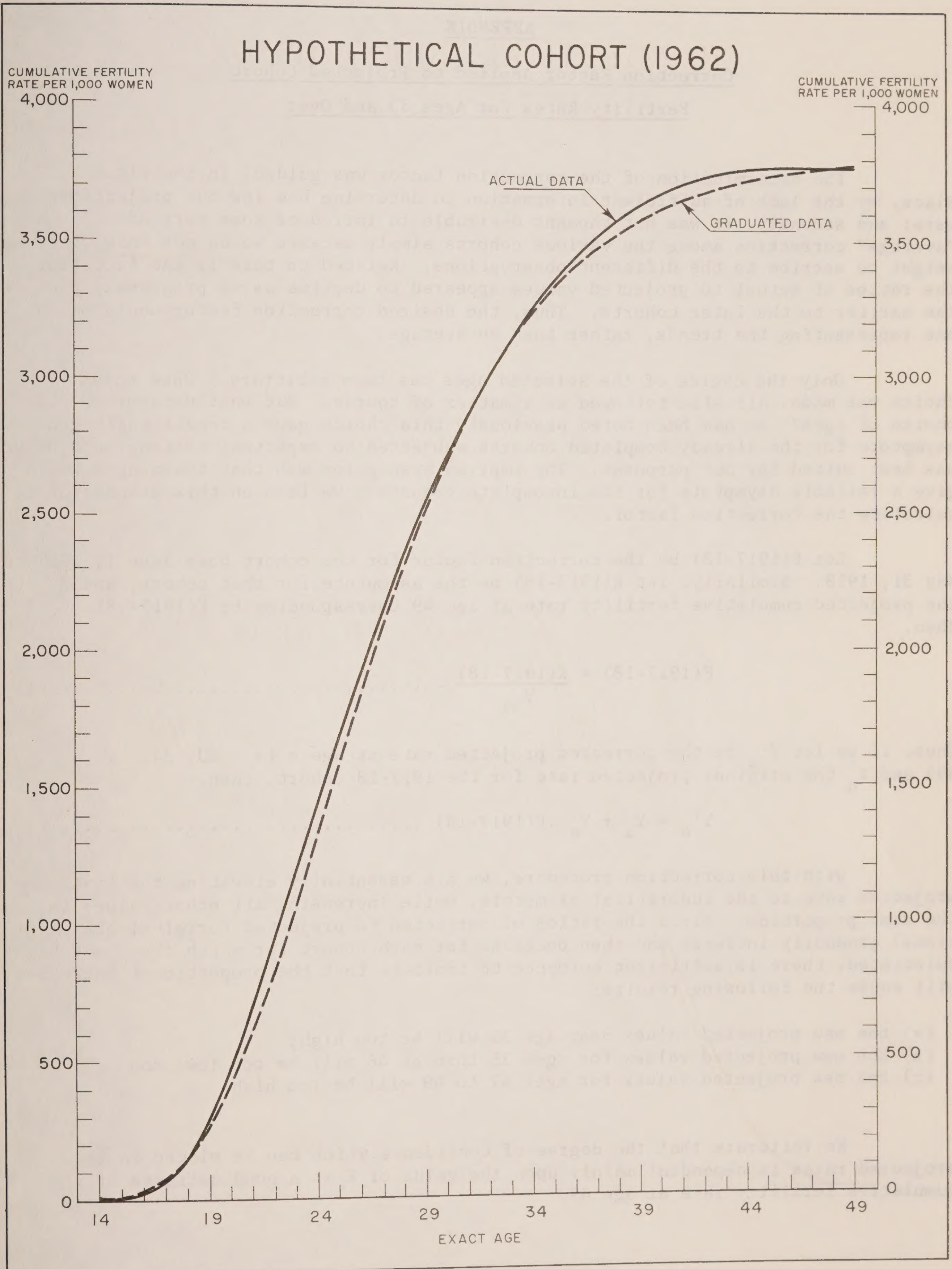
GRAPH A.3


1916-17 ACTUAL COHORT
CUMULATIVE FERTILITY RATE PER 1,000 WOMEN
4,000
3,500
3,000
2,500
2,000
1,500
1,000
500
0
CUMULATIVE FERTILITY RATE PER 1,000 WOMEN
ACTUAL DATA
GRADUATED DATA
14
19
24
29
34
39
44
49
EXACT AGE

GRAPH A.4


HYPOTHETICAL COHORT (1950)
CUMULATIVE FERTILITY RATE PER 1,000 WOMEN
CUMULATIVE FERTILITY RATE PER 1,000 WOMEN
4,000
3,500
3,000
2,500
2,000
1,500
1,000
500
0
ACTUAL DATA
GRADUATED DATA
14
19
24
29
34
39
44
49
EXACT AGE

GRAPH A.5


HYPOTHETICAL COHORT (1962)
CUMULATIVE FERTILITY RATE PER 1,000 WOMEN
CUMULATIVE FERTILITY RATE PER 1,000 WOMEN
4,000
3,500
3,000
2,500
2,000
1,500
1,000
500
0
ACTUAL DATA
GRADUATED DATA
14
19
24
29
34
39
44
49
EXACT AGE

APPENDIX

Correction Factor Applied to Projected Cohort Fertility Rates for Ages 33 and Over

The determination of the correction factor was guided, in the first place, by the lack of sufficient information to determine how low our projections were; and secondly, it was not thought desirable to introduce some sort of "average" correction among the various cohorts simply because we do not know what weight to ascribe to the different observations. Related to this is the fact that the ratios of actual to projected values appeared to decline as we progressed from the earlier to the later cohorts. Thus, the desired correction factor would be the one representing the trends, rather than an average.

Only the choice of the selected ages has been arbitrary. Once this choice was made, all else followed as a matter of course. But what determined our choice of ages? As has been noted previously this choice gave a consistently good asymptote for the already completed cohorts subjected to empirical testing, and hence was best suited for our purposes. The implied assumption was that these ages would give a reliable asymptote for the incomplete cohorts. We lean on this assumption to calculate the correction factor.

Let F(1917-18) be the correction factor for the cohort born June 1, 1917 to May 31, 1918. Similarly, let K(1917-18) be the asymptote for that cohort, and Y_{49} be the projected cumulative fertility rate at age 49 corresponding to K(1917-18). Then,

$$F(1917\text{-}18) = \frac{K(1917\text{-}18)}{Y_{49}} - 1 \qquad (1)$$

Thus, if we let Y'_a be the corrected projected rate at age a (a = 33, 34, ..., 49) and Y_a the original projected rate for the 1917-18 cohort, then,

$$Y'_a = Y_a + Y_a \cdot F(1917\text{-}18) \qquad (2)$$

With this correction procedure, we are essentially elevating the last projected rate to the theoretical asymptote, while increasing all other values in the same proportion. Since the ratios of corrected to projected (original projections) gradually increase and then decrease for each cohort for which they have been calculated, there is sufficient evidence to indicate that the proportional increases will cause the following results:

(a) the new projected values near age 33 will be too high;
(b) the new projected values for ages 36 through 46 will be too low; and
(c) the new projected values for ages 47 to 49 will be too high.

We reiterate that the degree of confidence which can be placed in the projected rates is dependent mainly upon the value of K as a good estimate of the cumulative fertility rate at age 49.